JOURNAL OF APPROXIMATION THEORY 70, 29–38 (1992)

Divergence Almost Everywhere of a Pointwise Comparison of Trigonometric Convolution Processes with Their Discrete Analogues

N. KIRCHHOFF* AND R. J. NESSEL

Lehrstuhl A für Mathematik, RWTH Aachen, Templergraben 55, D-5100 Aachen, Germany

Communicated by Paul L. Butzer

Received March 4, 1991

Continuing previous investigations this paper deals with negative results concerning a pointwise comparison of trigonometric convolution processes with their discrete analogues. Though the uniform errors of those processes are equivalent (under suitable conditions), application of an appropriate extension of a familiar lemma of A. P. Calderón in connection with a general quantitative resonance principle establishes that corresponding pointwise interpretations may fail almost everywhere.

1. INTRODUCTION

Let $C_{2\pi}$ be the Banach space of functions f, 2π-periodic and continuous on the real axis $\mathbf{R}$, endowed with the usual sup-norm $\|f\| := \sup\{|f(u)| : u \in \mathbf{R}\}$. Consider two sequences $\{S_n\}$, $\{T_n\}$ of bounded linear operators of $C_{2\pi}$ into itself, the norm of, e.g., T_n being denoted by $\|T_n\| := \sup\{\|T_n f\| : f \in C_{2\pi}, \|f\| \leqslant 1\}$. If the operators are polynomial and coincide on Π_n (set of trigonometric polynomials of degree $\leqslant n$) in the sense that for each $n \in \mathbf{N}$ (set of natural numbers)

$$S_n, T_n \colon C_{2\pi} \to \Pi_n, \qquad S_n p = T_n p \qquad \text{for all } p \in \Pi_n, \tag{1.1}$$

then it immediately follows (cf. [1]) that for each $f \in C_{2\pi}$, $n \in \mathbf{N}$

$$\frac{\|T_n f - f\|}{1 + \|S_n\| + \|T_n\|} \leqslant \|S_n f - f\| \leqslant (1 + \|S_n\| + \|T_n\|)\ \|T_n f - f\|.$$

If, moreover, the processes $\{S_n\}$, $\{T_n\}$ are equibounded, i.e.,

$$\|S_n\| = \mathcal{O}(1), \qquad \|T_n\| = \mathcal{O}(1) \qquad (n \to \infty), \tag{1.2}$$

* Supported by Deutsche Forschungsgemeinschaft Grant IIC4-Ne 171/7-2.

0021-9045/92 $5.00

then there exist constants $0 < C_1$, $C_2 < \infty$, independent of $f \in C_{2\pi}$, $n \in \mathbf{N}$, such that

$$C_1 \|T_n f - f\| \leqslant \|S_n f - f\| \leqslant C_2 \|T_n f - f\|. \tag{1.3}$$

Hence the assumptions (1.1, 2) imply that the uniform errors of the processes are indeed equivalent, apart from constants.

Turning to pointwise approximation, however, the situation may change completely. The present paper illustrates this (see also [5, 7]) in connection with approximation by trigonometric convolution operators and their discrete analogues. While the uniform comparison (1.3) holds true in this case, Section 3 shows that a corresponding pointwise interpretation fails almost everywhere, even for a Lipschitz continuous function. It may be mentioned that in [8] we were only able to establish this result on a denumerable set of points. Indeed, the present method of proof essentially depends on an appropriate extension of a lemma of A. P. Calderón as well as on a general quantitative resonance principle, already established in [7]. These tools are prepared in Section 2.

2. Tools

Let us begin with the following extension of a lemma of Calderón (cf. [15, p. 165]), a basic tool towards divergence almost everywhere (cf. [13, 14]). For $\alpha \in \mathbf{R}$, $M \subset \mathbf{R}$ we use the standard notations $\alpha M := \{\alpha x : x \in M\}$, $M + \alpha := \{x + \alpha : x \in M\}$ as well as $\|f\|_1 := \int_0^{2\pi} |f(u)|\, du$ for $f \in L^1_{2\pi}$, the space of 2π-periodic, Lebesgue integrable functions.

THEOREM 2.1. *Let H_k, $D_k \subset \mathbf{R}$ be (Lebesgue) measurable subsets such that H_k is 2π-periodic and D_k belongs to $[0, 2\pi]$ with Lebesgue measure $\lambda(D_k) \neq 0$ for each $k \in \mathbf{N}$. Suppose that*

$$\left\| \prod_{k=1}^{n} \left(1 - \frac{\lambda(D_k \cap (H_k - t))}{\lambda(D_k)}\right)\right\|_1 = o(1) \qquad (n \to \infty). \tag{2.1}$$

Then there exist points $y_k \in D_k$ such that $\limsup_{k \to \infty} (H_k - y_k) := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} (H_k - y_k)$ is a set of full measure.

Proof. The argument is essentially that employed for the proof of the standard assertion (see [15, p. 166]) which is concerned with the case $D_k = [0, 2\pi]$. Thus, with $\mathscr{C}A := \mathbf{R} \backslash A$, let $\chi_k(t)$ be the characteristic function of $\mathscr{C}H_k$ so that $\chi_k(t + y_k)$ is the one of $\mathscr{C}(H_k - y_k)$ for $y_k \in D_k$. By Fubini's theorem and the assumption (2.1) it follows that for $p_1 \in \mathbf{N}$

$$F(1, p_1) := \frac{1}{\lambda(D_1)} \int_{D_1} \frac{1}{\lambda(D_2)} \int_{D_2} \cdots \frac{1}{\lambda(D_{p_1})}$$

$$\times \int_{D_{p_1}} \left[\int_0^{2\pi} \prod_{k=1}^{p_1} \chi_k(t+y_k)\, dt \right] dy_{p_1} \cdots dy_2\, dy_1$$

$$= \int_0^{2\pi} \left[\prod_{k=1}^{p_1} \frac{1}{\lambda(D_k)} \int_{D_k} \chi_k(t+y_k)\, dy_k \right] dt$$

$$= \int_0^{2\pi} \left[\prod_{k=1}^{p_1} \frac{\lambda(D_k \cap \mathscr{C}(H_k - t))}{\lambda(D_k)} \right] dt$$

$$= \int_0^{2\pi} \left[\prod_{k=1}^{p_1} \left(1 - \frac{\lambda(D_k \cap (H_k - t))}{\lambda(D_k)} \right) \right] dt = o(1) \qquad (p_1 \to \infty).$$

Analogously one has $\lim_{p_{i+1} \to \infty} F(p_i + 1, p_{i+1}) = 0$ for every fixed $p_i \in \mathbb{N}$. Using this, the remaining part of the proof then indeed proceeds parallel to that of the standard assertion (see [15, p. 166]). ∎

Let us mention that the formulation of the lemma of Calderón is usually given in terms of a condition of type

$$\sum_{k=1}^{\infty} \frac{\lambda(D_k \cap (H_k - t))}{\lambda(D_k)} = \infty \qquad \text{a.e.,}$$

which is sufficient for

$$\lim_{n \to \infty} \prod_{k=1}^{n} \left(1 - \frac{\lambda(D_k \cap (H_k - t))}{\lambda(D_k)} \right) = 0 \qquad \text{a.e.}$$

But the latter condition is indeed equivalent to (2.1).

On the basis of Theorem 2.1 the divergence assertions mentioned then result from an application of the following quantitative resonance principle (see [7]).

For a Banach space X (with norm $\|\cdot\|$) let X^* be the set of non-negative, sublinear, and bounded functionals T on X, i.e., T maps X into $[0, \infty)$ such that for all $f, g \in X$ and scalars α

$$T(f+g) \leqslant Tf + Tg, \qquad T(\alpha f) = |\alpha|\, Tf,$$
$$\|T\|_{X^*} := \sup\{Tf\colon \|f\| \leqslant 1\} < \infty.$$

Let ω be an abstract modulus of continuity, thus a function, continuous on $[0, \infty)$, with

$$0 = \omega(0) < \omega(s) \leqslant \omega(s+t) \leqslant \omega(s) + \omega(t) \qquad (s, t > 0),$$

additionally satisfying

$$\lim_{t \to 0+} \frac{\omega(t)}{t} = \infty. \tag{2.2}$$

Let $\sigma(t)$ be a function, (strictly) positive on $(0, \infty)$, and $\{\varphi_n\}$ be a sequence, (strictly) decreasing with $\lim_{n \to \infty} \varphi_n = 0$. In these terms one has

THEOREM 2.2. *Let A, B be arbitrary index sets. Suppose that for families of functionals $\{U_t : t \in (0, \infty)\}$, $\{V_{n,\alpha} : n \in \mathbf{N},\ \alpha \in A\}$, $\{W_{n,\alpha} : n \in \mathbf{N},\ \alpha \in A\} \subset X^*$ with*

$$\|V_{n,\alpha}\|_{X^*} + \|W_{n,\alpha}\|_{X^*} \leqslant C_1 \qquad (n \in \mathbf{N},\ \alpha \in A) \tag{2.3}$$

there exist testelements $\{g_{n,\beta} : n \in \mathbf{N}, \beta \in B\} \subset X$ such that

$$\|g_{n,\beta}\| \leqslant C_2 \qquad (n \in \mathbf{N}, \beta \in B), \tag{2.4}$$

$$U_t g_{n,\beta} \leqslant C_3 \min\{1, \sigma(t)/\varphi_n\} \qquad (t \in (0, \infty), \beta \in B, n \in \mathbf{N}), \tag{2.5}$$

$$V_{n,\alpha} g_{j,\beta} + W_{n,\alpha} g_{j,\beta} = \mathcal{O}_j(\varphi_n) \qquad (\alpha \in A,\ \beta \in B, n \to \infty). \tag{2.6}$$

Moreover, for each subsequence $\{n_j\} \subset \mathbf{N}$ let there exist a sequence $\{M_k\}$ of subsets of A (more exactly $M_{\{n_j\},k}$), sequences of points $\{\beta_k\} \subset B$ and of numbers $\{\varepsilon_k\} \subset \mathbf{R}$ with $\lim_{k \to \infty} \varepsilon_k = 0$, and a constant $C_4 > 0$ such that for $\alpha \in M_k$

$$V_{n_k,\alpha} g_{n_k,\beta_k} \geqslant C_4 - \varepsilon_k, \tag{2.7}$$

$$W_{n_k,\alpha} g_{n_k,\beta_k} \leqslant \varepsilon_k. \tag{2.8}$$

Then for each modulus ω satisfying (2.2) *there exist a subsequence $\{n_j\}$ and a counterexample $f_\omega \in X$ with*

$$U_t f_\omega = \mathcal{O}(\omega(\sigma(t))) \qquad (t \to 0+), \tag{2.9}$$

$$V_{n,\alpha} f_\omega \neq o(\omega(\varphi_n)) \qquad (n \to \infty), \tag{2.10}$$

$$V_{n,\alpha} f_\omega \neq \mathcal{O}(W_{n,\alpha} f_\omega) \qquad (n \to \infty), \tag{2.11}$$

simultaneously for each $\alpha \in M_{\{n_j\}} := \limsup_{k \to \infty} M_{\{n_j\},k}$.

This quantitative version of a resonance principle looks rather technical, but in view of the many parameters occurring it is indeed very flexible for applications. For a proof see [7] and the literature cited there. Here let us continue with some remarks explaining roughly how this result contributes to the present problem: First of all, Theorem 2.2 indeed delivers the negative result (2.11) on the comparison of the processes V and W, and it is given in quantitative terms inasmuch as (2.9) assures a certain smoothness of the counterexample f_ω, whereas (2.10) may be interpreted as a precision of its nonsmoothness. If A is equal to a point set of $\mathbf{R}$, the families of functionals V and W usually represent the pointwise remainders of certain approximation processes. Therefore the index sets A (and B) should be arbitrary, not only denumerable. The proof of Theorem 2.2 proceeds via a suitable gliding hump technique which delivers the candidate

$$f_\omega = \sum_{j=1}^{\infty} \omega(\varphi_{n_j})\, g_{n_j, \beta_j}. \tag{2.12}$$

Indeed, in view of the properties of ω and $\{\varphi_n\}$ one may first successively select the strictly increasing subsequence $\{n_j\}$ which additionally may be assumed to be of the form

$$n_1 = 4, \qquad n_{k+1} = \tfrac{1}{2}[(4s_k+1)(2n_k+1)-1] \tag{2.13}$$

for some $s_k \in \mathbf{N}$. For this subsequence there then exist, by assumption, sets $M_k \subset A$ and points $\beta_k \in B$ such that (2.7, 2.8) hold true. Thus it is essential that the assumptions around (2.7, 2.8) be satisfied for each subsequence. Our candidate for a counterexample being given via the infinite series (2.12), it is almost obvious that additional properties of the testelements $g_{n,\beta}$ may transfer to f_ω. For example, if all the $g_{n,\beta}$ are real-valued functions, then f_ω will be real-valued, too. The result then is that the assertions hold true on a limes superior of certain abstract sets. In other words, the general theory finally delivers a condensation of singularities on a limes superior of index sets. It is in this connection that Theorem 2.1 assures this limes superior to be a set of full measure.

3. Divergence Almost Everywhere of a Pointwise Comparison

With regard to typical representatives for sequences of operators satisfying (1.1), the present paper deals with trigonometric convolution operators and their discrete analogues.

For an even, polynomial kernel of degree $n \in \mathbf{N}$, given by

$$\chi_n(x) := \sum_{k=-n}^{n} \rho_{k,n} e^{ikx} \tag{3.1}$$

with $\rho_{-k,n} = \rho_{k,n}$, $\rho_{0,n} = 1$, and for $f \in C_{2\pi}$ let the trigonometric convolution operator be defined by

$$F_n f(x) := \frac{1}{2\pi} \int_0^{2\pi} f(u)\, \chi_n(x-u)\, du \tag{3.2}$$

and its discrete analogue by

$$J_n f(x) := \frac{1}{2n+1} \sum_{j=0}^{2n} f(u_{j,n}) \chi_n(x - u_{j,n}), \tag{3.3}$$

where $u_{j,n} = 2\pi j/(2n+1)$, $0 \leqslant j \leqslant 2n$. Note that (3.3) may be interpreted either as a quadrature formula for the integral (3.2) (cf. [3, 4]) or as a linear mean of the interpolation polynomial, associated with the knots $\{u_{j,n}\}$ (cf. [2, 6, 9–11, 12, p. 413ff]). For $h_k(x) := e^{ikx}$, $k \in \mathbf{Z}$ (set of integers), one has

$$F_n h_k(x) = \rho_{k,n} h_k(x) = J_n h_k(x) \qquad (|k| \leqslant n), \tag{3.4}$$

and additionally

$$F_n h_k(x) = 0, \qquad J_n h_k(x) = 1 \qquad (|k| = 2n+1). \tag{3.5}$$

Moreover, if one of the processes $\{F_n\}$ or $\{J_n\}$ is equibounded, then so is the other one (cf. [9, 10]). Hence (1.1, 2) are fulfilled if, e.g.,

$$\|\chi_n\|_1 = \mathcal{O}(1) \qquad (n \to \infty) \tag{3.6}$$

holds true, which yields the equivalence (1.3) of the uniform errors of the processes.

If one is interested in a pointwise interpretation of (1.3), it is quite obvious that in the present setting one also has

$$C_1 |F_n f(x) - f(x)| \leqslant |J_n f(x) - f(x)| \leqslant C_2 |F_n f(x) - f(x)|$$

for smooth functions, e.g., for polynomials (cf. (3.4)). The situation changes, however, if the functions are less smooth. Indeed, Theorem 3.1 delivers the existence of a counterexample $f_0 \in C_{2\pi}$ such that

$$\limsup_{n \to \infty} \frac{|J_n f_0(x) - f_0(x)|}{|F_n f_0(x) - f_0(x)|} = \infty \tag{3.7}$$

for almost every $x \in \mathbf{R}$. Furthermore, Theorem 3.1 even ensures a certain smoothness of the counterexample f_0 which now will be measured in terms of the rth modulus of continuity ($r \in \mathbf{N}$)

$$\omega_r(f, t) := \sup \left\{ \left\| \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} f(x + kh) \right\| : |h| \leqslant t \right\}.$$

For any abstract modulus of continuity ω and $r \in \mathbf{N}$ Lipschitz classes are then defined by

$$\mathrm{Lip}_r\, \omega := \{ f \in C_{2\pi} : \omega_r(f, t) = \mathcal{O}(\omega(t^r)),\ t \to 0+ \}.$$

THEOREM 3.1. *Let* $\{\chi_n : n \in \mathbf{N}\}$ *be a sequence of even, polynomial kernels* (3.1), *satisfying* (3.6) *and*

$$1-\rho_{j,n}=\mathcal{O}_j(n^{-r}) \qquad (j\in\mathbf{N},\, n\to\infty) \tag{3.8}$$

for some $r\in\mathbf{N}$. *Then for each modulus of continuity* ω *satisfying* (2.2) *there exists a* (*real-valued*) *counterexample* $f_\omega \in \mathrm{Lip}_r\,\omega$ *such that* $(n\to\infty)$

$$|J_n f_\omega(x)-f_\omega(x)| \neq o(\omega(n^{-r})), \tag{3.9}$$

$$|J_n f_\omega(x)-f_\omega(x)| \neq \mathcal{O}(|F_n f_\omega(x)-f_\omega(x)|) \tag{3.10}$$

simultaneously for almost every $x\in\mathbf{R}$.

Proof. To apply Theorem 2.2 set $X=C_{2\pi}$, $A=B=\mathbf{R}$, $U_t f=\omega_r(f,t)$, $\sigma(t)=t^r$, $\varphi_n=n^{-r}$, $\varepsilon_k=\pi/2(k+1)$, $g_{n,y}(x)=\cos((2n+1)(x+y))$,

$$V_{n,x} f=|J_n f(x)-f(x)|, \qquad W_{n,x} f=|F_n f(x)-f(x)|.$$

Since $\|\chi_n\|_1\leqslant C$, one has $\|W_{n,x}\|_{X^*}\leqslant C^*<\infty$ and $\|V_{n,x}\|_{X^*}\leqslant C^*$ (cf. [9–11]), thus (2.3). Moreover, $\|g_{n,y}\|=1$ and

$$U_t g_{n,y}\leqslant \min\{2^r\,\|g_{n,y}\|,\, t^r\,\|g_{n,y}^{(r)}\|\}\leqslant C\min\{1,\sigma(t)/\varphi_n\}.$$

In view of (3.4, 3.8) one also obtains (2.6). Now let $\{n_j\}\subset\mathbf{N}$ be an arbitrary subsequence satisfying (2.13). Let

$$\begin{aligned} H_k&=\bigcup_{j\in\mathbf{Z}}\frac{\pi}{2(2n_k+1)}\left[2j+1-\frac{1}{k+1},\, 2j+1+\frac{1}{k+1}\right],\\ D_k&=[0,2\pi]\cap\bigcup_{j\in\mathbf{Z}}\frac{\pi}{2(2n_k+1)}\left[2j-\frac{1}{2},\, 2j+\frac{1}{2}\right], \end{aligned} \tag{3.11}$$

and set $M_k=H_k-y_k$ for some $y_k\in D_k$, still to be chosen appropriately. If $x\in H_k-y_k$, then by (3.4, 3.5)

$$\begin{aligned} V_{n_k,x}\,g_{n_k,y_k}&=|\cos((2n_k+1)\,y_k)-\cos((2n_k+1)(x+y_k))|\\ &\geqslant|\cos((2n_k+1)\,y_k)|-|\cos((2n_k+1)(x+y_k))|\\ &\geqslant\frac{1}{\sqrt{2}}-\frac{\pi}{2}\,\frac{1}{k+1}=:C_4-\varepsilon_k,\\ W_{n_k,x}\,g_{n_k,y_k}&=|\cos((2n_k+1)(x+y_k))|\leqslant\frac{\pi}{2}\,\frac{1}{k+1}=\varepsilon_k. \end{aligned}$$

Hence conditions (2.7, 2.8) hold as well, and Theorem 2.2 delivers a counterexample $f_\omega\in\mathrm{Lip}_r\,\omega$ satisfying (3.9, 3.10) simultaneously on $H=\limsup_{k\to\infty}(H_k-y_k)$. It remains to show that there exist appropriate points $y_k\in D_k$ such that H is a set of full measure.

To apply Theorem 2.1, first note that $\lambda(D_k)=\pi$. Let us consider the function

$$f_k(t) := 1 - \frac{\lambda(D_k \cap (H_k - t))}{\lambda(D_k)} = \frac{1}{\pi}\,\lambda(D_k \cap \mathscr{C}(H_k - t)). \tag{3.12}$$

In view of (3.11) one has $f_k(t)=f_k(t+\pi/(2n_k+1))$, in particular $f_k \in L^1_{2\pi}$. Moreover, the estimate

$$f_k(t) \leqslant 1 - \frac{1}{k+1} \qquad \text{for} \quad t \in \frac{\pi}{2n_k+1}\left[\frac{1}{4}, \frac{3}{4}\right] \tag{3.13}$$

holds. Indeed, since (cf. (3.11)) $\mathscr{C}(H_k - t)$ equals

$$\bigcup_{j\in\mathbf{Z}} \frac{1}{2n_k+1}\left(\pi j - \frac{\pi}{2} - (2n_k+1)t + \frac{\pi}{2(k+1)},\ \pi j + \frac{\pi}{2} - (2n_k+1)t - \frac{\pi}{2(k+1)}\right),$$

one has for

$$t \in \frac{\pi}{2n_k+1}\left[\frac{1}{4}, \frac{1}{4} + \frac{1}{2(k+1)}\right]$$

(analogously for the interval $\pi/(2n_k+1)[3/4 - 1/2(k+1), 3/4]$)

$$D_k \cap \mathscr{C}(H_k - t) \subset$$

$$[0, 2\pi] \cap \bigcup_{j\in\mathbf{Z}} \frac{1}{2n_k+1}\left[\pi j - \frac{\pi}{4},\ \pi j + \frac{\pi}{2} - (2n_k+1)t - \frac{\pi}{2(k+1)}\right]$$

whereas for

$$t \in \frac{\pi}{2n_k+1}\left[\frac{1}{4} + \frac{1}{2(k+1)}, \frac{3}{4} - \frac{1}{2(k+1)}\right]$$

there holds true

$$D_k \cap \mathscr{C}(H_k - t) \subset$$

$$[0, 2\pi) \cap \bigcup_{j\in\mathbf{Z}} \left\{\left[\pi j - \frac{\pi}{4},\ \pi j + \frac{\pi}{2} - (2n_k+1)t - \frac{\pi}{2(k+1)}\right] \cup \left[\pi(j+1) - \frac{\pi}{2} - (2n_k+1)t + \frac{\pi}{2(k+1)},\ \pi j + \frac{\pi}{4}\right]\right\},$$

which then already implies (3.13). Let us introduce the abbreviations $d_k := 2(2n_k+1)$ and

$$A_k := \bigcup_{l\in\mathbf{Z}} \frac{2\pi}{d_k}\left\{l + \left[\frac{1}{4}, \frac{3}{4}\right]\right\}, \qquad B_k := \bigcup_{l\in\mathbf{Z}} \frac{2\pi}{d_k}\left\{l + \left[\frac{3}{4}, \frac{5}{4}\right]\right\}.$$

By (2.13) one then has $d_k = b_{k-1} d_{k-1}$ with $b_{k-1} = 4s_{k-1} + 1 \geqslant 4$, and in view of (3.13) and the definition of f_k

$$f_k(t) \leqslant \begin{cases} a_k := 1 - \dfrac{1}{k+1}, & t \in A_k \\ 1, & t \in B_k. \end{cases} \tag{3.14}$$

It remains to show that $\|\prod_{k=1}^{n} f_k\|_1 = o(1)$, thus (2.1).

To this end let $k \in \mathbf{N}$ be arbitrary, fixed. Then the interval $[\pi/4, 9\pi/4]$ is divided into d_k subintervals of length π/d_k, contained in A_k (in view of (2.13) consider $l = n_k/2 \in \mathbf{N}$) where $f_k(t) \leqslant a_k$, and d_k subintervals, contained in B_k where $f_k(t) \leqslant 1$. Now consider the partition of $[\pi/4, 9\pi/4]$ by intervals of A_{k-1} and B_{k-1}. Since $d_k = b_{k-1} d_{k-1}$ with $b_{k-1} \geqslant 4$, one has at most $d_k/2 + 2d_{k-1}$ intervals of length π/d_k where $f_{k-1} f_k$ takes values less than 1, a_{k-1}, a_k, and $a_{k-1} a_k$, respectively. Note that the additional terms $2d_{k-1}$ result from those subintervals of length π/d_k which belong to different subintervals of length π/d_{k-1}. Thus there are at most

$$\frac{d_k}{2^{k-1}} + \frac{d_{k-1}}{2^{k-3}} + \cdots + \frac{d_3}{2} + d_2 + 2d_1$$

intervals of length π/d_k, where $\prod_{j=1}^{k} f_j$ takes values less than 1, a_1, a_2, $a_1 a_2$, ..., $a_1 \cdots a_k$, respectively. Therefore it follows that

$$\begin{aligned}
\left\| \prod_{j=1}^{k} f_j \right\|_1 &= \int_{\pi/4}^{9\pi/4} f_1(t) \cdots f_k(t)\, dt \\
&\leqslant \frac{\pi}{d_k} \left(\frac{d_k}{2^{k-1}} + \frac{d_{k-1}}{2^{k-3}} + \cdots + 2d_1 \right) \prod_{j=1}^{k} (1 + a_j) \\
&= \frac{\pi}{d_k} \left(\frac{d_k}{2^{k-1}} + \frac{d_k}{2^{k-3} b_{k-1}} + \frac{d_k}{2^{k-4} b_{k-1} b_{k-2}} + \cdots + \frac{2d_k}{b_{k-1} \cdots b_1} \right) \\
&\quad \times \prod_{j=1}^{k} \left(2 - \frac{1}{j+1} \right) \\
&= \pi \left(2 + 2^2 \left[\frac{2}{b_{k-1}} + \frac{2}{b_{k-1}} \frac{2}{b_{k-2}} + \cdots + \frac{2}{b_{k-1}} \cdots \frac{2}{b_1} \right] \right) \\
&\quad \times \prod_{j=1}^{k} \left(1 - \frac{1}{2(j+1)} \right) \\
&\leqslant \pi \left(2 + 4 \left[\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{k-1}} \right] \right) \prod_{j=1}^{k} \left(1 - \frac{1}{2(j+1)} \right) \\
&\leqslant 6\pi \prod_{j=1}^{k} \left(1 - \frac{1}{2(j+1)} \right) = o(1),
\end{aligned}$$

since $\sum_{j=1}^{\infty} j^{-1} = \infty$. ∎

Let us mention that in [8] the existence of a counterexample $f_1 \in C_{2\pi}$ was shown such that also (cf. (3.7))

$$\limsup_{n \to \infty} \frac{|F_n f_1(x) - f_1(x)|}{|J_n f_1(x) - f_1(x)|} = \infty \tag{3.15}$$

on a denumerable set of points $x \in \mathbf{R}$, in fact on a dense set of second category. It is then tempting to conjecture the existence of a counterexample $f_1 \in C_{2\pi}$ such that (3.15) holds true even on a set of full measure. So far, however, we have not been able to specify the parameters in such a way that this result would follow as an application of Theorems 2.1 and 2.2.

REFERENCES

1. D. L. BERMAN, Some remarks on the problem of speed of convergence of polynomial operators, *Izv. Vyssh. Uchebn. Zaved. Mat.* **5**, No. 24 (1961), 3–5. [In Russian]
2. S. N. BERNSTEIN, Sur l'interpolation trigonométrique par la méthode des moindres carrés, *C. R. Acad. Sci. URSS (N.S. 5)* **4** (1934), 1–8.
3. R. BOJANIC AND O. SHISHA, Approximation of continuous, periodic functions by discrete linear positive operators, *J. Approx. Theory* **11** (1974), 231–235.
4. R. DEVORE AND J. SZABADOS, Saturation theorems for discretized linear operators, *Anal. Math.* **1** (1975), 81–89.
5. W. DICKMEIS, R. J. NESSEL, AND E. VAN WICKEREN, The divergence almost everywhere of a pointwise comparison of Fejér and Abel–Poisson means, *Bull. London Math. Soc.* **18** (1986), 606–608.
6. F. ESSER AND E. GÖRLICH, Diskrete und kontinuierliche Summationsverfahren von Orthogonalreihen, *in* "Mathematical Structures—Computational Mathematics—Mathematical Modelling" (B. Sendov, Ed.), pp. 235–244, Publ. House Bulg. Acad. Sci., Sofia, 1975.
7. N. KIRCHHOFF AND R. J. NESSEL, Convolution processes of Fejér's type and the divergence almost everywhere of a pointwise comparison, to appear.
8. N. KIRCHHOFF AND R. J. NESSEL, Some negative results concerning a pointwise comparison of a trigonometric convolution process with its discrete analogue, *in* "Approximation Theory" (Proc. Conf. Kecskemét (1990)), in press.
9. S. M. LOSINSKI, On an analogy between the summation of Fourier series and that of interpolation trigonometric polynomials, *C.R. Acad. Sci. URSS (N.S.)* **39** (1943), 83–87.
10. S. M. LOSINSKI, On convergence and summability of Fourier series and interpolation processes, *Mat. Sb.* **14**, No. 56 (1944), 175–268.
11. J. MARCINKIEWICZ, Sur l'interpolation I, II, *Studia Math.* **6** (1936), 1–17, 67–81.
12. I. P. NATANSON, "Konstruktive Funktionentheorie", Akademie-Verlag, Berlin, 1955.
13. K. I. OSKOLKOV, An estimate of the rate of approximation of a continuous function and its conjugate by Fourier sums on a set of full measure, *Izv. Akad. Nauk SSSR Ser. Mat.* **38** (1974), 1393–1407 [in Russian]; English translation *Math. USSR—Izv.* **8** (1974), 1372–1386.
14. E. M. STEIN, On limits of sequences of operators, *Ann. of Math.* **74** (1961), 140–170.
15. A. ZYGMUND, "Trigonometric Series, Vol. II," Cambridge Univ. Press, Cambridge, U.K., 1968.